The classification on simple Moufang loops

SANDU N. I.

Abstract

Let $C(F)$ be a matrix Cayley-Dickson algebra over field F. By $M_0(F)$ we denote the loop containing of all elements of algebra $C(F)$ with norm 1. It is shown in this paper that with precision till isomorphism the loops $M_0(F)/ < -1 >$ they and only they are simple non-associative Moufang loops, where F are subfields of algebraic closed field.

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The purpose of this paper is to classify the non-associative simple Moufang loops. The Moufang loop is simple if it has no non-trivial proper normal subloops, or equivalently, if it has no non-trivial proper homomorphic images. For basic definitions and properties of Moufang loops see [1].

It is well known that for an alternative algebra A with the unit element 1 the set $U(A)$ of all invertible elements of A forms a Moufang loop with respect to multiplication [2]. Let now $C(F)$ be a matrix Cayley-Dickson algebra over arbitrary field F, and let $M_0(F)$ be the set of all elements of $C(F)$ with norm 1. $M_0(F)$ is a normal subloop of $U(C(F))$. Let $Z(M_0(F))$ be the center of $M_0(F)$. Paige L. shows in [3, Theorem 4.1 and corollary to Lemma 3.4] that $M_0(F)/Z(M_0(F))$ is a simple, non-associative, Moufang loop and center $Z(M_0(F))$ is a group of order 2 if the characteristic of F is not 2; otherwise $Z(M_0(F)) = 1$. Further, using this result and the powerful apparatus of finite groups theories, in [4] Liebeck M. finalizes the classification of finite non-associative simple Moufang loops, started in [5, 6]. He shows that such loops are isomorphic to one of the loops $M(GF(q))$, where $GF(q)$ is a finite Galois field, modulo the center. Within this paper all non-associative simple Moufang loops are classified through methods of alternative algebras. With precision till isomorphism it is one of loops $M_0(C(F))/ < -1 >$, where $C(F)$ denotes a matrix Cayley-Dickson algebra $C(F)$ over a subfield F of algebraically closed field. Further, the needed results of alternative algebras from [7, 8] will be used without reference.

By analogy to Lemma 1 from [9] it is proved.

Lemma 1. *Let A be an alternative algebra and let Q be a subloop of $U(A)$. Then the restriction of any homomorphism of algebra A upon Q will be a homomorphism on the loop. More concretely, any ideal J of A induces a normal subloop $Q \cap (1 + J)$ of Q.*

Let L be a free Moufang loop, let F be a field and let FL be a loop algebra of loop L over field F. We remind that FL is a free module with basis $\{g|g \in L\}$ and the multiplication of elements of the basis is defined by their multiplication in loop L. Let $(u, v, w) = uv \cdot w - u \cdot vw$ denote the associator of elements u, v, w of algebra FL. We denote by I the ideal of loop algebra FL, generated by the set

$$\{(a, b, c) + (b, a, c), (a, b, c) + (a, c, b)|\forall a, b, c \in L\}.$$

It is shown in [9] that algebra FL/I is alternative and loop L is embedded (isomorphically) in the loop $U(FL/I)$. Further we identify the loop L with its isomorphic image in $U(FL/I)$. Hence the free loop L is a subloop of loop $U(FL/I)$. Without causing any misunderstandings, like in [9], we will denote by FL the quotient algebra FL/I and call it ″loop algebra″ (in inverted commas). Sums $\sum_{g \in L} \alpha_g g$, are elements of algebra FL, where $\alpha_g \in F$. Further, we will identify the field F with subalgebra $F1$ of algebra FL, where 1 is the unit of loop L.

Let now Q be an arbitrary Moufang loop. Then Q has a representation as a quotient loop L/H of the free Moufang loop L by the normal subloop H. We denote by ωH the ideal of ″loop algebra″ FL, generated by the elements $1-h$ $(h \in H)$. By Lemma 1 ωH induces a normal subloop $K = L \cap (1+\omega H)$ of loop L and $F(L/K) = FL/\omega H$.

We denote $L/K = \overline{Q}$, thus $FL/\omega H = F\overline{Q}$. As every element in FL is a finite sum $\sum_{g \in L} \alpha_g g$, where $\alpha_g \in F$, $g \in L$, then the finite sum $\sum_{q \in \overline{Q}} \alpha_q q$, where $\alpha_q \in F$, $q \in \overline{Q}$ will be elements of algebra $F\overline{Q}$. Let us determine the homomorphism of F-algebras $\varphi : FL \to F(L/H)$ by the rule $\varphi(\sum \lambda_q q) = \sum \lambda_q Hq$. The mapping φ is F-linear, then for $h \in H, q \in L$ we have $\varphi((1 - h)q) = Hq - H(hq) = Hq - Hq = 0$. Hence $\omega H \subseteq \mathrm{ker}\varphi$. The loop $\overline{Q}$ is a subloop of loop $U(F\overline{Q})$ and as $\omega H \subseteq \mathrm{ker}\varphi$, then the homomorphisms $FL \to FL/\omega H = F\overline{Q}$ and $FL \to FL/\mathrm{ker}\varphi = F(L/H) = FQ$ induces a homomorphism π of loop $\overline{Q}$ upon loop Q. Hence we have.

Lemma 2. *Let Q be an arbitrary Moufang loop. Then the loop $\overline{Q}$ is embedded in loop of invertible elements $U(F\overline{Q})$ of alternative algebra $F\overline{Q}$ and the homomorphism $L \to FL/\omega H$ of″loop algebra″ FL induces a homomorphism $\pi : \overline{Q} \to Q$ of loops.*

Let now Q be a simple Moufang loop. Then $\ker\pi$ will be a proper maximal normal subloop of $\overline{Q}$. Let J_1, J_2 be proper ideals of algebra $F\overline{Q}$. We prove that the sum $J_1 + J_2$ is also a proper ideal of $F\overline{Q}$. Indeed, by Lemma 1 $K_1 = \overline{Q} \cap (1 + J_1)$, $K_2 = \overline{Q} \cap (1 + J_2)$ will be normal subloops of loop $\overline{Q}$. We have that $K_1 \subseteq \ker\pi$, $K_2 \subseteq \ker\pi$. Then product $K_1K_2 \subseteq \ker\pi$, as well. But $K_1K_2 = (\overline{Q} \cap (1 + J_1))(\overline{Q} \cap (1 + J_2)) = \overline{Q} \cap (1 + J_1)(1 + J_2) = \overline{Q} \cap (1 + J_1 + J_2 + J_1J_2) = \overline{Q} \cap (1 + J_1 + J_2)$. Hence $\overline{Q} \cap (1 + J_1 + J_2) \subseteq \ker\pi$, i.e. $\overline{Q} \cap (1 + J_1 + J_2)$ is a proper normal subloop of $\overline{Q}$. Then from Lemma 1 it follows that $J_1 + J_2$ is a proper ideal of algebra $F\overline{Q}$, as required.

We denote by S the ideal of algebra $F\overline{Q}$, generated by all proper ideals J_i $(i \in I)$ of $F\overline{Q}$. Let us show that S is also a proper ideal of algebra $F\overline{Q}$. If I is a finite set, then the statement follows from first case. Let us now consider the second possible case. The algebra $F\overline{Q}$ is generated as a F-module by elements $x \in \overline{Q}$. Let there be such ideals $J_1, \ldots, J_k$ that for element $1 \neq a \in \overline{Q}$ $a \in \sum J_i$ and let us suppose that for element $b \in \overline{Q}$ $b \notin \sum J_i$. We denote by T the set of all ideals of algebra $F\overline{Q}$, containing the element a, but not containing the element b. By Zorn's Lemma there is a maximal ideal I_1 in T. We denote by I_2 the ideal of algebra $F\overline{Q}$, generated by all proper ideals of $F\overline{Q}$ that don't belong to ideal I_1. Then $S = I_1 + I_2$. I_1, I_2 are proper ideals of $F\overline{Q}$ and by first case S is also proper ideal of $F\overline{Q}$. By Lemma 1 $K = \overline{Q} \cap (1 + S)$ is a normal subloop of $\overline{Q}$. We denote $\overline{\overline{Q}} = \overline{Q}/K$. Then $F\overline{\overline{Q}} = F\overline{Q}/S$ is a simple algebra. As $K \subseteq \ker\pi$ then π induce a homomorphism $\rho : \overline{\overline{Q}} \to Q$. Hence we prove.

Lemma 3. *Let Q be a simple non-associative Moufang loop. Then $F\overline{\overline{Q}}$ is a simple alternative algebra and the homomorphism $\pi : \overline{Q} \to Q$ induces a homomorphism $\rho : \overline{\overline{Q}} \to Q$.*

Let F be an arbitrary field. Let us consider a classical matrix Cayley-Dickson algebra $C(F)$. It consists of matrices of form $\begin{pmatrix} \alpha_1 & \alpha_{12} \\ \alpha_{21} & \alpha_2 \end{pmatrix}$, where $\alpha_1, \alpha_2 \in F$, $\alpha_{12}, \alpha_{21} \in F^3$. The addition and multiplication by scalar of elements of algebra $C(F)$ is represented by ordinary addition and multiplication by scalar of matrices, and the multiplication of elements of algebra $C(F)$ is defined by the rule

$$\begin{pmatrix} \alpha_1 & \alpha_{12} \\ \alpha_{21} & \alpha_2 \end{pmatrix} \begin{pmatrix} \beta_1 & \beta_{12} \\ \beta_{21} & \beta_2 \end{pmatrix} =$$

$$\begin{pmatrix} \alpha_1\beta_1 + (\alpha_{12}, \beta_{21}) & \alpha_1\beta_{12} + \beta_2\alpha_{12} - \alpha_{21} \times \beta_{21} \\ \beta_1\alpha_{21} + \alpha_2\beta_{21} + \alpha_{12} \times \beta_{12} & \alpha_2\beta_2 + (\alpha_{21}, \beta_{12}) \end{pmatrix},$$

where for vectors $\gamma = (\gamma_1, \gamma_2, \gamma_3)$, $\delta = (\delta_1, \delta_2, \delta_3) \in A^3$ $(\gamma, \delta) = \gamma_1\delta_1 + \gamma_2\delta_2 + \gamma_3\delta_3$ denotes their scalar product and $\gamma \times \delta = (\gamma_2\delta_3 - \gamma_3\delta_2, \gamma_3\delta_1 - \gamma_1\delta_3, \gamma_1\delta_2 -$

$\gamma_2\delta_1)$ denotes the vector product. Algebra $C(F)$ is alternative. It is also quadratic over F, i.e. each element $a \in C(F)$ satisfies the identity

$$a^2 - t(a)a + n(a) = 0, n(a), t(a) \in F$$

and admits composition, i.e.

$$n(ab) = n(a)n(b)$$

for $a, b \in C(F)$. Track $t(a)$ and norm $n(a)$ are defined by the equalities $t(a) = \alpha_1 + \alpha_2$, $n(a) = \alpha_1\alpha_2 - (\alpha_{12}, \alpha_{21})$.

We have $M_0(F) = \{u \in C(F) | n(u) = 1\}$. Further, $n(1) = 1$, and it follows from the relations $n(ab) = n(a)n(b)$, $n(\alpha a) = \alpha^2 n(a)$ that $-1 \in M_0(F)$. Obviously -1 belongs to the center of algebra $C(F)$. Then -1 belongs to the center $Z(M_0(F)$ of loop $M_0(F)$. Therefore the subloop $< -1 >$, generated by element -1, is normal in $M_0(F)$ and from Paige's results [3], presented at the beginning of the article it follows.

Lemma 4. *Let F be an arbitrary field. Then the Moufang loop $M(F) = M_0(F)/ < -1 >$ of the matrix Cayley-Dickson algebra $C(F)$ is simple and the loop $< -1 >$ coincide with center of loop $M_0(F)$.*

By Lemma 4 the center Z of loop $M_0(F)$ coincides with subloop $< -1 >$. As $M_0(F)/Z$ is a simple loop, a question appears. Is the center Z of loop $M_o(F)$ emphasized by the direct factor? The answer is negative. Let field F consist of 5 elements and let H be a direct completion of center Z. If α were the generator of the multiplicative group of field F, then one of the elements $\pm \begin{pmatrix} \frac{1}{\alpha} & 0 \\ 0 & \alpha \end{pmatrix}$ would lie in H. The square of this element is equal to -1, i.e., it lies in the intersection $H \cap Z$, which is impossible. Therefore center Z cannot have a direct factor in $M_0(F)$.

Let now P be an algebraically closed field and let Q be a simple non-associative Moufang loop. By Lemma 3 the loop $\overline{\overline{Q}}$ is embedded in loop of invertible elements of simple alternative algebra $P\overline{\overline{Q}}$. We denote $\overline{\overline{Q}} = G$.

If $a \in G$, then it follows from the equality $aa^{-1} = 1$ that $n(a)n(a)^{-1} = 1$, i.e. $n(a) \neq 0$. Associator (a, b, c) of elements a, b, c of an arbitrary loop is defined by the equality $ab \cdot c = (a \cdot bc)(a, b, c)$. Identity $(xy)^{-1} = y^{-1}x^{-1}$ holds in Moufang loops. Therefore, if a, b, c are elements of Moufang loop G, then $u = (a, b, c) = (a \cdot bc)^{-1})(ab \cdot c) = (c^{-1}b^{-1} \cdot a^{-1})(ab \cdot c)$, $n(u) = n(c^{-1})n(b^{-1})n(a^{-1})n(a) \cdot n(b)n(c) = n(c)^{-1}n(b)^{-1}n(a)^{-1}n(a)n(b)n(c) = 1$, i.e. $u \in M_0(P)$. We denote by G' the subloop generated by all associators of Moufang loop G. If $G' = G$, then $G \subseteq M_0(P)$, i.e. the loop G is embedded in $M_0(P)$. Now we suppose that $G' \neq G$. It is shown in [10, 11] that the subloop G' is normal in G. The finite sum $\sum_{g \in G} \alpha_g g$, where $\alpha_g \in P$, $g \in G$

are elements of algebra PG. Let $\eta : PG \to P(G/G')$ be a homomorphism of P-algebras determined by rule $\eta(\sum \alpha_g g) = \sum \alpha_g gG'$ $(g \in G)$ and let $P(G/G') = PG/\mathrm{ker}\eta$. As the quotient loop $P(G/G')$ is non-trivial, then $PG/\mathrm{ker}\eta \neq PG$. Hence $\mathrm{ker}\eta$ is a proper ideal of PG. The algebra PG is simple. Then the ideal $\mathrm{ker}\eta$ cannot be the proper ideal of PG. Hence the case $G' \neq G$ is impossible and, consequently, the loop G is embedded in loop $M_0(P)$.

The alternative algebra PG is simple. By Kleinfeld Theorem [12, see also 7, 8] it is a Cayley-Dickson algebra over their center. Field P is algebraically closed. Then algebra PG is split. The matrix Cayley-Dickson algebra $C(P)$ is also split. But any two split non-associative composition algebras over an algebraically closed field are isomorphic. Therefore algebra $PG = P\overline{\overline{Q}}$ is isomorphic to the matrix Cayley-Dickson algebra $C(P)$.

If M is a F-module, and N is its subset, then the denotation $F\{N\}$ means the F-submodule generated by N. We denote $P\overline{\overline{Q}} = C_P(\overline{\overline{Q}})$. Consequently, it is proved.

Lemma 5. *Let P be an algebraically closed field and let Q be a simple Moufang loop. Then loop $\overline{\overline{Q}}$ is embedded in split Cayley-Dickson algebra $C_P(\overline{\overline{Q}}) = P\{\overline{\overline{Q}}\}$.*

Let P be an algebraically closed field and let H be a subloop of loop $M_0(P)$. Let $a_{ij} = (a_{ij}^{(1)}, a_{ij}^{(2)}, a_{ij}^{(3)})$. If the matrices elements $\alpha_i, \alpha_{ij}^{(k)}$ of all matrices $\begin{pmatrix} \alpha_1 & \alpha_{12} \\ \alpha_{21} & \alpha_2 \end{pmatrix} \in H$ generate the subfield F of field P, then we will say that H is a *loop over field F*. If H is strictly contained into $M_0(F)$, then we say that H is a *proper over field F subloop* of loop $M_0(F)$.

Lemma 6. *Let F be an arbitrary field. Then the Moufang loop $M_0(F)$ doesn't contain proper over F non-associative subloops of type $\overline{\overline{Q}}$, considered in Lemma 5.*

Proof. Let P be an algebraic closing of field F. Let us suppose the contrary, that loop $M_0(F) = L$ contains a proper over F non-associative subloop $H \subset L$ of type $\overline{\overline{Q}}$. Let $P\{H\}$, $P\{L\}$ are the matrix Cayley-Dickson algebras, considered in Lemma 5. We consider the subalgebras $C_F(H) = F\{H\} \subseteq P\{H\}$, $C_F(L) = F\{L\} \subseteq P\{L\}$, defined in Lemma 3. $C_F(H)$ and $C_F(L)$ are matrix Cayley-Dickson algebras and $C_F(H)$ is a non-associative subalgebra of $C_F(L)$. The algebras $C_F(H)$, $C_F(L)$ are isomorphic as split non-associative composition algebras over the same field F.

By the supposition, $H \subset L$. Then it follows from the isomorphism of composition algebras $F\{L\}$ and $F\{H\}$ that element $1 \neq a \in L \backslash H$ is linearly

expressed through the elements of loop H in algebra $F\{H\} = C_F(H)$. Let FH be a loop algebra (without inverted commas) of loop H. It follows from the definition of $''$the loop algebra$''$ $C_F(H)$ that $C_F(H) = FH/I$, where I is the ideal of loop algebra (without inverted commas) FH [9]. It follows from here that in loop algebra FH element $a \in L\backslash H$ is linearly expressed through the elements of loop H. Further, $H \subset L$, therefore $FH \subseteq FL$. Then in loop algebra FL element $a \in L$ is linearly expressed through the elements of loop $H \subset L$. But this contradicts the definition of loop algebra FL, which is a free F-module with basis consisting of elements of loop L. Consequently, the simple Moufang loop $M(F)$ has no proper over field F non-associative subloops. This completes the proof of Lemma 6.

Theorem 1. *Let P be an algebraically closed field. Only and only the loops $M(F)$ of the matrix Cayley-Dickson algebra $C(F)$, where F is a subfield of field P, are with precise till isomorphism non-associative simple Moufang loops. Loop $M(F)$ is quotient loop $M_0(F)/ < -1 >$, where $M_0(F)$ consists of all elements of $C(F)$ with norm 1, and the subloop $< -1 >$, generated by element -1, coincide with the center of loop $M_0(F)$.*

Proof. If F is an arbitrary subfield of P then by Lemma 4 the loop $M(F)$ is a simple non-associative Moufang loop. Let now Q be an arbitrary non-associative simple Moufang loop. By Lemma 3 the loop $\overline{\overline{Q}}$ is embedded in loop $M_0(P)$. We identify $\overline{\overline{Q}}$ with isomorphic image in $M_0(P)$. Let loop $\overline{\overline{Q}}$ be presented by matrices $a = \begin{pmatrix} \alpha_1 & \alpha_{12} \\ \alpha_{21} & \alpha_2 \end{pmatrix}$ and let F be a subfield of field P, generated by all matrices elements of matrices a. Loop $\overline{\overline{Q}}$ is a loop over field F. By Lemma 6 there is only one non-associative Moufang loop over field F, and namely $M_0(F)$. Therefore $\overline{\overline{Q}} = M_0(F)$. By Paige's results [3] (presented at the beginning of the article) the loop $M_0(F)$ posed only one homomorphism: $M_0(F) \to M_0(F)/ < -1 >= M(F)$. Then the homomorphism $\rho : \overline{\overline{Q}} \to Q$ coincides with this homomorphism. Hence $Q = M(F)$. This completes the proof of Theorem 1.

It is worth mentioning that in [13] a particular case of Theorem 1 is proved through other means.

It is known that the field of complex numbers is algebraically closed and contains as subfields all finite fields. Then from Theorem 1 there follows the main result of article [4] about the classification of finite non-associative simple Moufang loops, conducted with the help of the finite groups theory.

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TIRASPOL STATE UNIVERSITY OF MOLDOVA

The author's home address:
Sandu Nicolae Ion
Deleanu str 1,

Apartment 60
Kishinev MD-2071, Moldova
E-mail: sandumn@yahoo.com